ADITYA ENGINEERING COLLEGE (A)

PARTIAL DIFFERENTIAL EQUATIONS
&
VECTOR CALCULUS

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Unit-I

SPECIAL FUNCTIONS.

Unit-II

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VECTOR INTEGRATION.

BETA AND GAMMA FUNCTIONS

BETA AND GAMMA FUNCTIONS

Beta Function

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m>0$, $n>0$ is defined as the ***beta function*** and it is denoted by $\beta(m, n)$.

$\therefore$ $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx,\ m>0,\ n>0$$

m, n are called the parameters of beta function.

This integral is also known as the first Eulerian integral.

Symmetric Property of Beta Function

$$\beta(m,n) = \beta(n,m)$$

The beta function is symmetric with respect to its parameters.

Proof

We have $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Put $x = 1 - y$ in (1), then $1 - x = y$ and $dx = -dy$

When $x = 0, y = 1$ and when $x = 1, y = 0$

$$\therefore \qquad \beta(m, n) = \int_1^0 (1-y)^{m-1} y^{n-1}(-dy)$$

$$= -\int_1^0 y^{n-1}(1-y)^{m-1}dy = \int_0^1 y^{n-1}(1-y)^{m-1}dy = \beta(n, m)$$

Different Forms of Beta Function

(1) Beta Function is an Improper Integral

That is
$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

Proof

By definition,
$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Put $x = \dfrac{y}{1+y}$ $\quad \therefore \quad dx = \dfrac{(1+y)\cdot 1 - y\cdot 1}{(1+y)^2}dy = \dfrac{1}{(1+y)^2}dy$

Now $x = \dfrac{y}{1+y} \Rightarrow (1+y)x = y \Rightarrow x + xy = y \Rightarrow y(x-1) = -x \Rightarrow y = \dfrac{x}{1-x}$

When $x = 0, y = 0$ and when $x = 1, y = \infty$

$$\therefore \qquad \beta(m,n)=\int_0^\infty\left(\frac{y}{1+y}\right)^{m-1}\left(1-\frac{y}{1+y}\right)^{n-1}\frac{dy}{(1+y)^2}$$

$$=\int_0^\infty\frac{y^{m-1}(1+y-y)^{n-1}}{(1+y)^{m-1+n-1+2}}dy=\int_0^\infty\frac{y^{m-1}}{(1+y)^{m+n}}dy$$

$$\therefore \qquad \beta(m,n)=\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx$$

This integral is an improper integral of the first kind.

(2) Beta Function Interms of Trignometric Function

$$\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta d\theta$$

Proof

By definition $\qquad \beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Put $\qquad x=\sin^2\theta \quad \therefore \quad dx=2\sin\theta\cos\theta d\theta$

Put $x = \sin^2\theta \quad \therefore \quad dx = 2\sin\theta\cos\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{2}$

$$\therefore \quad \beta(m,n) = \int_0^{\pi/2} (\sin^2\theta)^{m-1}(1-\sin^2\theta)^{n-1} 2\sin\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-2}\theta\cos^{2n-2}\theta\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow \quad \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

Note In some practical problems we come across definite integrals involving trignometric functions which can be evaluated interms of beta functions.

We have $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$

$$\Rightarrow \quad \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \frac{1}{2}\beta(m,n)$$

If $p = 2m-1 \quad \Rightarrow \quad 2m = p+1 \quad \Rightarrow \quad m = \frac{p+1}{2}$

and $q = 2n-1 \quad \Rightarrow \quad 2n = q+1 \quad \Rightarrow \quad n = \frac{q+1}{2}$

$$\therefore \quad \int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

4) $B(m,n) = B(m+1,n) + B(m,n+1)$

Soln:- we have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

consider $B(m+1,n) + B(m,n+1) = \int_0^1 x^{(m+1)-1}(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^{(n+1)-1}\,dx$

$$= \int_0^1 x^{m-1}\cdot x\,(1-x)^{n-1}\,dx + \int_0^1 x^{m-1}(1-x)^{n-1}\cdot(1-x)\,dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\left[x+1-x\right]dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

$$= B(m,n)$$

$\therefore B(m,n) = B(m+1,n) + B(m,n+1)$.

5. If m and n are positive integers then

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

6. a) $B(m,1) = \frac{1}{m}$ b) $B(1,n) = \frac{1}{n}$.

a) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$B(m,1) = \int_0^1 x^{m-1}(1-x)^{1-1}\,dx$$

$$= \int_0^1 x^{m-1}\,dx = \left.\frac{x^{(m-1)+1}}{(m-1)+1}\right]_0^1$$

$$= \left.\frac{x^m}{m}\right]_0^1 = \frac{1}{m}$$

$$\therefore B(m,1) = \frac{1}{m}$$

b) $B(1,n) = \int_0^1 x^{1-1}(1-x)^{n-1}\,dx$

$= \int_0^1 (1-x)^{n-1}\,dx$

$= \left[\dfrac{(1-x)^{(n-1)+1}}{[(n-1)+1](-1)}\right]_0^1$

$= \left[\dfrac{(1-x)^n}{-n}\right]_0^1$

$= 0 + \dfrac{1}{n} = \dfrac{1}{n}$

$\therefore B(1,n) = \dfrac{1}{n}.$

7. $B(1,1) = 1$

THE GAMMA FUNCTION

The integral $\int_0^\infty e^{-x}x^{n-1}dx\ (n>0)$ is defined as the **gamma function** with parameter n and it is denoted by Γn.

$$\therefore \quad \Gamma n = \int_0^\infty e^{-x}x^{n-1}dx, \ (n>0)$$

This integral is also known as Euler's integral of the second kind.

Properties of Gamma Function

(1) Prove that $\Gamma 1 = 1$.

Proof

By definition,
$$\Gamma 1 = \int_0^\infty e^{-x}x^{1-1}dx = \int_0^\infty e^{-x}dx = \left[\frac{e^{-x}}{-1}\right]_0^\infty = -[0-1] = 1$$

(2) Prove that $\Gamma(n+1) = n\Gamma n$

Proof

By definition $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$

$$\therefore \quad \Gamma(n+1) = \int_0^\infty e^{-x} x^{n+1-1} dx$$

$$= \int_0^\infty e^{-x} x^n dx$$

$$= \left[x^n \frac{e^{-x}}{-1}\right]_0^\infty - \int_0^\infty n x^{n-1} \frac{e^{-x}}{-1} dx = n\int_0^\infty e^{-x} x^{n-1} dx = n\Gamma n$$

$$\therefore \quad \Gamma(n+1) = n\Gamma n$$

This is true for all positive values of n.

(3) If n is an Integer ≥ 1, then $\Gamma n = (n-1)!$

We have $\Gamma(n+1) = n\Gamma n$

$$\Gamma n = (n-1)\Gamma(n-1)$$
$$= (n-1)(n-2)\Gamma(n-2)$$
$$= (n-1)(n-2)\ldots 3\cdot 2\cdot 1\cdot \Gamma 1$$

$\Rightarrow$ $$\Gamma n = (n-1)(n-2)\ldots 3\cdot 2\cdot 1 = (n-1)!$$

(4) Prove that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof

By definition, $$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x} x^{1/2-1} dx = \int_0^\infty e^{-x} x^{-1/2} dx$$

Put $x = y^2$ $\quad\therefore\quad dx = 2ydy$

When $x = 0$, $y = 0$ and when $x = \infty$, $y = \infty$

$\therefore$ $$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-y^2} y^{-1}\cdot 2ydy = 2\int_0^\infty e^{-y^2} dy$$

Now $$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2} dy \quad \text{and} \quad \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx$$

$\therefore$ $$\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2} dx \cdot 2\int_0^\infty e^{-y^2} dy$$

$\Rightarrow$ $$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} dxdy$$ [since the limits are constants]

Put $x = r\cos\theta$, and $y = r\sin\theta$

$\therefore$ $r^2 = x^2 + y^2$ and $\tan\theta = \frac{y}{x}$

and $dxdy = rdrd\theta$ [$\because$ Jacobian value $= r$]

When x, y varies from 0 to ∞, r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$

$$\therefore \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta = 4\left[\int_0^{\pi/2} d\theta\right]\left[\int_0^{\infty} e^{-r^2} r\,dr\right]$$

$$= 4[\theta]_0^{\pi/2}\int_0^{\infty} e^{-r^2} r\,dr = 4\cdot\frac{\pi}{2}\int_0^{\infty} e^{-r^2} r\,dr = 2\pi\int_0^{\infty} e^{-r^2} r\,dr$$

Let $r^2 = u$ $\therefore$ $2rdr = du \Rightarrow rdr = \frac{du}{2}$

When $r = 0, u = 0$ and when $r = \infty, u = \infty$

$$\therefore \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 2\pi\int_0^{\infty} e^{-u}\frac{du}{2} = \pi\int_0^{\infty} e^{-u} du$$

$$= \pi\left[\frac{e^{-u}}{-1}\right]_0^{\infty} = -\pi[e^{-\infty} - e^0] = -\pi(0-1) = \pi$$

$$\therefore \Gamma(1/2) = \sqrt{\pi}$$

Relation between Beta and Gamma Functions

Prove that $\beta(m, n) = \dfrac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}$, $m > 0$, $n > 0$

Proof

By definition, $\Gamma m = \int_0^\infty e^{-t} t^{m-1} dt$

Let $t = x^2 \quad \therefore \quad dt = 2x dx$

When $t = 0$, $x = 0$ and when $t = \infty$, $x = \infty$

$\therefore$ $$\Gamma m = \int_0^\infty e^{-x^2} x^{2m-2} \cdot 2x dx = 2\int_0^\infty e^{-x^2} x^{2m-1} dx \quad (1)$$

Similarly, $$\Gamma n = 2\int_0^\infty e^{-y^2} y^{2n-1} dy \quad (2)$$

$\therefore$ (1) × (2) ⇒ $\Gamma m \cdot \Gamma n = 2\int_0^\infty e^{-x^2} x^{2m-1} dx \cdot 2\int_0^\infty e^{-y^2} y^{2n-1} dy$

⇒ $\Gamma m \cdot \Gamma n = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1} dx\, dy$ (3)

[since the limits are constants]

Changing to polar coordinates by putting

$$x = r\cos\theta, \quad y = r\sin\theta$$

we get $r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}$ and $dxdy = rdrd\theta$

When x, y varies from 0 to ∞, r varies from 0 to ∞ and θ varies from 0 to $\frac{\pi}{2}$

$\therefore$ (3) becomes

$$\Gamma m \cdot \Gamma n = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} (r\cos\theta)^{2m-1} (r\sin\theta)^{2n-1} rdrd\theta$$

$$= 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r^{2m+2n-2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta d\theta \cdot \int_0^\infty e^{-r^2} r^{2m+2n-1} dr$$

$$= 2\beta(n, m) \cdot \int_0^\infty e^{-r^2} r^{2m+2n-1} dr$$

Let $$I = \int_0^\infty e^{-r^2} r^{2m+2n-1} dr$$

Put $t = r^2$ $$\therefore \quad dt = 2r dr \Rightarrow dr = \frac{dt}{2r} = \frac{dt}{2t^{1/2}}$$

When $r = 0, t = 0$ and when $r = \infty, t = \infty$

$$\therefore \quad I = \int_0^\infty e^{-t} \left(t^{1/2}\right)^{2m+2n-1} \frac{dt}{2t^{1/2}} = \frac{1}{2}\int_0^\infty e^{-t} t^{m+n-1/2} \cdot t^{-1/2} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t} t^{m+n-1} dt = \frac{1}{2}\Gamma(m+n) \qquad \text{[by definition]}$$

$$\therefore \quad (4) \Rightarrow \quad \Gamma m \cdot \Gamma n = 2\beta(n, m) \cdot \frac{\Gamma(m+n)}{2}$$

$$\Rightarrow \quad \Gamma m \cdot \Gamma n = \beta(m, n)\Gamma(m+n) \qquad [\because \quad \beta(m, n) = \beta(n, m)]$$

$$\therefore \quad \beta(m, n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}$$

- Find the values of

1. $\Gamma(7/2)$ 2. $\Gamma(-3/2)$

Sol: 1)

$$\Gamma(7/2) = 5/2\,\Gamma(5/2) \quad (\because \Gamma(n+1) = n\Gamma(n))$$

$$= 5/2 \times 3/2\,\Gamma(3/2)$$

$$= 5/2 \times 3/2 \times 1/2\,\Gamma(1/2)$$

$$= \frac{15}{8}\sqrt{\pi}$$

2) $$\Gamma(-3/2)=\frac{\Gamma(-3/2+1)}{-3/2}\qquad (\because \Gamma(n+1)=n\Gamma(n)) \Rightarrow \Gamma(n)=\frac{\Gamma(n+1)}{n})$$

$$=\frac{\Gamma(-1/2+1)}{-3/2\times -1/2}$$

$$=\frac{4}{3}\Gamma(1/2)=\frac{4}{3}\sqrt{\pi}$$

Similarly,

$$\Gamma(-1/2)=-2\sqrt{\pi}$$

Beta Function

Definition

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called beta function and is denoted by $B(m,n)$

i.e.,

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$$

where m>0, n>0

Gamma Function

Definition

The definite integral $\int_0^\infty e^{-x}x^{n-1}dx$ is called Gamma function and is denoted by $\Gamma(n)$

i.e.,

$$\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx \text{ where } n>0$$

Note:(Read the proofs)

1. $B(m,n) = B(n,m)$

2. $B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

3. $\int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} B(\frac{p+1}{2}, \frac{q+1}{2})$

4. $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx$

5. $\Gamma(1) = 1$

6. $\Gamma(n) = (n-1)\Gamma(n-1)$

7. $\Gamma(n) = (n-1)!$ If n is a non negative integer

8. $\Gamma(1/2) = \sqrt{\pi}$

9. Relation between Beta and Gamma functions:

$$B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

10. $\Gamma n \Gamma(1-n) = \dfrac{\pi}{\sin n\pi}$

Find the value of $\int_0^1 x^7(1-x)^6\,dx$.

Solution.

We know $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Now $\int_0^1 x^7(1-x)^6dx=\int_0^1 x^{8-1}(1-x)^{7-1}dx=\beta(8,7)=\frac{\Gamma 8\cdot\Gamma 7}{\Gamma(8+7)}=\frac{7!\cdot 6!}{14!}$

Evaluate $\int_0^1 x^{11}(1-x)^5\,dx$.

Solution.

We know $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

Now $\int_0^1 x^{11}(1-x)^5dx=\int_0^1 x^{12-1}(1-x)^{6-1}dx=\beta(12,6)=\frac{\Gamma 12\cdot\Gamma 6}{\Gamma(12+16)}=\frac{11!\cdot 5!}{27!}$

Evaluate $\int_0^1 \frac{x}{\sqrt{(1-x^5)}}\,dx$

Sol: Given,

$$\int_0^1 \frac{x}{\sqrt{(1-x^5)}}\,dx = \int_0^1 x(1-x^5)^{-1/2}\,dx$$

Put $t = x^5 \Rightarrow x = t^{1/5} \Rightarrow dx = \frac{1}{5}t^{-4/5}dt$

Limits: If x=0 then t=0,

x=1 then t=1

$$\int_0^1 \frac{x}{\sqrt{(1-x^5)}}\,dx = \int_0^1 x(1-x^5)^{-1/2}\,dx$$

$$\int_0^1 \frac{x}{\sqrt{(1-x^5)}}dx = \int_0^1 x(1-x^5)^{-1/2}dx$$

$$= \int_0^1 t^{1/5}(1-t)^{-1/2}\frac{1}{5}t^{-4/5}dt$$

$$= \frac{1}{5}\int_0^1 t^{-3/5}(1-t)^{-1/2}dt$$

$$= \frac{1}{5}\int_0^1 t^{\frac{2}{5}-1}(1-t)^{\frac{1}{2}-1}dt$$

$$= \frac{1}{5} B(\frac{2}{5}, \frac{1}{2})$$

$$= \frac{1}{5} \frac{\Gamma(2/5)\Gamma(1/2)}{\Gamma(\frac{2}{5} + \frac{1}{2})}$$

$$== \frac{1}{5} \frac{\sqrt{\pi}\Gamma(2/5)}{\Gamma(9/10)}$$

3. $$\int_0^3 \frac{dx}{\sqrt{(9-x^2)}}$$

Sol: Given,

$$\int_0^3 \frac{dx}{\sqrt{(9-x^2)}} dx = \int_0^3 (9-x^2)^{-1/2} dx = \frac{1}{3}\int_0^3 (1-x^2/9)^{-1/2} dx$$

Put $t = \dfrac{x^2}{9} \Rightarrow x = 3t^{1/2} \Rightarrow dx = \dfrac{3}{2} t^{-1/2} dt$

Limits: If x=0 then t=0,

x=3 then t=1

Hence, $\int_0^3 \frac{dx}{\sqrt{(9-x^2)}} = \frac{1}{3}\int_0^3 (1-x^2/9)^{-1/2}dx$

$$= \frac{1}{3}\int_0^1 (1-t)^{-1/2}\frac{3}{2}t^{-1/2}dt$$

$$= \frac{1}{2}\int_0^1 t^{-1/2}(1-t)^{-1/2}dt$$

$$= \frac{1}{2}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{\frac{1}{2}-1}dt$$

$$= \frac{1}{2} B(\frac{1}{2}, \frac{1}{2})$$

$$= \frac{1}{2} \frac{\Gamma(1/2)\Gamma(1/2)}{\Gamma(\frac{1}{2}+\frac{1}{2})}$$

$$= \frac{1}{2} \frac{\pi}{\Gamma(1)} = \frac{1}{2}\pi$$

4. $$4\int_0^\infty \frac{x^2}{1+x^4}\,dx$$

Sol: Given, $$4\int_0^3 \frac{x^2}{1+x^4}\,dx$$

We have, $$B(m,n)=\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

Put $t = x^4 \Rightarrow x = t^{1/4} \Rightarrow dx = \frac{1}{4}t^{-3/4}dt$

Limits: If x=0 then t=0,

x= ∞ then t= ∞

Hence, $4\int_0^\infty \frac{x^2}{1+x^4}dx = 4\int_0^\infty \frac{t^{2/4}}{1+t}\frac{1}{4}t^{-3/4}dt$

$$= \int_0^\infty \frac{t^{-1/4}}{1+t}dt \qquad [\because B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

$$= B(\frac{3}{4},\frac{1}{4})$$

here $m-1 = \frac{-1}{4} \Rightarrow m = \frac{3}{4}$

$m+n = 1 \Rightarrow n = 1 - \frac{3}{4} = \frac{1}{4}$]

$$= \frac{\Gamma(3/4)\Gamma(1/4)}{\Gamma(\frac{3}{4}+\frac{1}{4})}$$

$$= \frac{\Gamma(\frac{1}{4})\Gamma(1-\frac{1}{4})}{\Gamma(1)}$$

$$= \frac{\pi}{\sin \pi / 4}$$

$$= \frac{\pi}{1/\sqrt{2}}$$

$$= \sqrt{2}\pi$$

VI Evaluate the following integrals using B-Γ functions.

1. $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}\,dx$

2. $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

② Solution :-

$$\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = \int_0^\infty \frac{(x^8-x^{14})}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}\,dx$$

$$= B(9,15) - B(15,9)$$

$$\left[\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = B(m,n)\right]$$

$= B(15,9) - B(15,9) \quad [\because B(m,n) = B(n,m)]$

$= 0.$

$$\therefore \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx = 0.$$

VII (1) Evaluate $4\cdot\int_0^\infty \frac{x^2}{1+x^4}\,dx$ using B & Γ functions.

Solution:- Let $x^2 = \tan\theta \Rightarrow \theta = \tan^{-1}x^2$

$x = \sqrt{\tan\theta}$

$dx = \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta\cdot d\theta$

$x = 0, \quad \theta = \tan^{-1}x^2 = \tan^{-1}0 = 0.$

$x \to \infty \quad \theta = \tan^{-1}\infty = \frac{\pi}{2}$

4. $\int_0^2 x(8-x^3)^{1/3}\,dx$

Soln: Let $x^3 = 8y \Rightarrow y = \frac{x^3}{8}$

$x = (8y)^{1/3}$

$= (2^3 y)^{1/3} = 2y^{1/3}$

$\therefore x = 2y^{1/3}$

$dx = 2 \cdot \frac{1}{3} y^{\frac{1}{3}-1}\, dy$

$dx = \frac{2}{3} y^{-\frac{2}{3}}\, dy$

$x = 0, \quad y = \frac{x^3}{8} = \frac{0}{8} = 0.$

$x = 2, \quad y = \frac{2^3}{8} = \frac{8}{8} = 1$

$$\int_0^2 x(8-x^3)^{1/3}\,dx = \int_0^1 2y^{1/3}\cdot(8-8y)^{1/3}\cdot\frac{2}{3}\,y^{-\frac{2}{3}}\,dy$$

$$= \frac{4}{3}\int_0^1 y^{\frac{1}{3}-\frac{2}{3}}\left[8(1-y)\right]^{1/3}dy$$

$$= \frac{4}{3}\int_0^1 y^{-\frac{1}{3}}(2^{3})^{1/3}(1-y)^{1/3}dy$$

$$= \frac{4}{3}\cdot 2\int_0^1 y^{-\frac{1}{3}}(1-y)^{1/3}dy$$

$$m-1 = -\frac{1}{3} \Rightarrow m = -\frac{1}{3}+1 = \frac{2}{3} \qquad \boxed{\therefore m = \frac{2}{3}}$$

$$n-1 = \frac{1}{3} \Rightarrow n = \frac{1}{3}+1 = \frac{4}{3} \qquad \boxed{\therefore n = \frac{4}{3}}$$

$$= \frac{8}{3}\int_0^1 y^{\frac{2}{3}-1}(1-y)^{\frac{4}{3}-1}dy$$

$$= \frac{8}{3} B\left(\frac{2}{3}, \frac{4}{3}\right)$$

$$\left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{8}{3} \cdot \frac{\Gamma\left(\frac{2}{3}\right) \cdot \Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)}$$

$$= \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right) \Gamma\left(\frac{1}{3}+1\right)}{\Gamma(2)}$$

$$\left[\because \Gamma(n+1) = n\Gamma(n)\right]$$

$$= \frac{8}{3} \cdot \frac{\Gamma\left(\frac{2}{3}\right) \cdot \frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{\Gamma(1+1)}$$

$$\left[\because \Gamma(n+1) = n!\right]$$

$$= \frac{8}{3} \frac{1}{3} \cdot \frac{\Gamma\left(1-\frac{1}{3}\right) \Gamma\left(\frac{1}{3}\right)}{1!}$$

$$\left[\because \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{8}{9} \cdot \frac{\pi}{\sin\frac{\pi}{3}}$$

$$= \frac{8\pi}{9} \cdot \frac{1}{\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{8\pi}{9} \cdot \frac{2}{\sqrt{3}} = \frac{16\pi}{9\sqrt{3}}.$$

$$\therefore \int_0^2 x(8-x^3)^{1/3}\,dx = \frac{16\pi}{9\sqrt{3}}$$

5. $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$

Soln:- $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \int_0^1 (1-x^4)^{-\frac{1}{2}}\,dx$

Let $x^4 = y$

$x = y^{1/4}$

6. Prove that $\int_0^1 (1-x^n)^{1/n}\, dx = \frac{1}{n} \frac{\left(\Gamma\left(\frac{1}{n}\right)\right)^2}{2\Gamma\left(\frac{2}{n}\right)}$.

solⁿ:-

Let $x^n = y$

$x = y^{\frac{1}{n}}$

$dx = \frac{1}{n} \cdot y^{\frac{1}{n}-1}\, dy$

$x = 0, \quad y = x^n = 0$

$x = 1, \quad y = 1^n = 1.$

$\int_0^1 (1-x^n)^{\frac{1}{n}}\, dx = \int_0^1 (1-y)^{\frac{1}{n}} \cdot \frac{1}{n} y^{\frac{1}{n}-1}\, dy$

$= \frac{1}{n} \int_0^1 y^{\left(\frac{1}{n}\right)-1} (1-y)^{\frac{1}{n}}\, dy$

$$= \frac{1}{n}\int_0^1 y^{\frac{1}{n}-1}(1-y)^{\left(\frac{1}{n}+1\right)-1}\,dy$$

$$= \frac{1}{n}\cdot B\left(\frac{1}{n}, \frac{1}{n}+1\right)$$

$$= \frac{1}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\cdot\Gamma\left(\frac{1}{n}+1\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{n}+1\right)} \qquad \left[\because B(m,n)=\frac{\Gamma(m)\cdot\Gamma(n)}{\Gamma(m+n)}\right]$$

$$= \frac{1}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\cdot\frac{1}{n}\Gamma\left(\frac{1}{n}\right)}{\frac{2}{n}\Gamma\left(\frac{2}{n}\right)} \qquad \left[\because \Gamma(n+1)=n\Gamma(n),\ \therefore \Gamma\left(\frac{2}{n}+1\right)=\frac{2}{n}\cdot\Gamma\left(\frac{2}{n}\right)\right]$$

$$= \frac{1}{n^2}\cdot\frac{n}{2}\cdot\frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\Gamma\left(\frac{2}{n}\right)}$$

$$= \frac{1}{2n} \frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{\Gamma\left(\frac{2}{n}\right)}$$

$$\therefore \int_0^1 (1-x^n)^{1/n}\, dx = \frac{1}{n} \cdot \frac{\left[\Gamma\left(\frac{1}{n}\right)\right]^2}{2 \cdot \Gamma\left(\frac{2}{n}\right)}$$

7. $\int_0^1 \frac{dx}{(1-x^3)^{1/3}}$

8. $\int_0^1 x^3 \sqrt{1-x}\, dx$

9. $\int_0^1 x^4 (1-x)^2\, dx$

10. $\int_0^1 x^5 (1-x)^3\, dx$.

Evaluate $\int_0^{\pi/2} \sin^8\theta \cos^7\theta d\theta$.

Solution.

We know

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\therefore \int_0^{\pi/2} \sin^8\theta \cos^7\theta d\theta = \frac{1}{2}\beta\left(\frac{8+1}{2}, \frac{7+1}{2}\right) \qquad [p = 8, q = 7]$$

$$= \frac{1}{2}\beta\left(\frac{9}{2}, 4\right) = \frac{1}{2}\frac{\Gamma\left(\frac{9}{2}\right)\Gamma 4}{\Gamma\left(\frac{9}{2}+4\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{9}{2}\right)\times 3!}{\Gamma\left(\frac{17}{2}\right)} = \frac{1\cdot 2\cdot 3\cdot \Gamma\left(\frac{9}{2}\right)}{2\cdot\frac{15}{2}\cdot\frac{13}{2}\cdot\frac{11}{2}\cdot\frac{9}{2}\Gamma\left(\frac{9}{2}\right)} = \frac{16}{15\cdot 13\cdot 11\cdot 3} = \frac{16}{6435}$$

Evaluate $\int_0^{\pi/2} \sin^5 \theta d\theta$.

Solution.

We know that

$$\int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\therefore \quad \int_0^{\pi/2} \sin^5 \theta d\theta = \int_0^{\pi/2} \sin^5 \theta \cos^0 \theta d\theta = \frac{1}{2}\beta\left(\frac{5+1}{2}, \frac{0+1}{2}\right) \qquad [p=5, q=0]$$

$$= \frac{1}{2}\beta\left(3, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma 3 \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{6+1}{2}\right)} = \frac{1}{2}\frac{2! \cdot \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{7}{2}\right)} = \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{8}{15}$$

Evaluate $\int_0^{\pi/2} \cos^8 \theta d\theta$.

Solution.

We know that $\int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$

$$\therefore \quad \int_0^{\pi/2} \cos^8 \theta d\theta = \int_0^{\pi/2} \sin^0 \theta \cos^8 \theta d\theta \qquad [p = 0, q = 8]$$

$$= \frac{1}{2}\beta\left(\frac{0+1}{2}, \frac{8+1}{2}\right) = \frac{1}{2}\beta\left(\frac{1}{2}, \frac{9}{2}\right)$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{9}{2}\right)}{\Gamma\left(\frac{9}{2}+\frac{1}{2}\right)}$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{1}{2}\right)\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma 5} = \frac{1}{2^5}\frac{7\cdot 5\cdot 3\pi}{1\cdot 2\cdot 3\cdot 4} = \frac{35\pi}{2^8} = \frac{35\pi}{256}$$

Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta.$

Solution.
We know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Now, $$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^{-1/2}\theta d\theta$$

$$= \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{-\frac{1}{2}+1}{2}\right) \qquad \left[\because\ p = \frac{1}{2}, q = -\frac{1}{2}\right]$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{4}\right)$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\tan\theta}\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)}{\Gamma 1} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right).$$

Evaluate $\int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}} d\theta.$

Solution.

We know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\therefore \quad \int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}} d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta d\theta$$

$$= \frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\tan\theta}} d\theta = \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right)}{\Gamma 1} = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{3}{4}\right).$$

Evaluate $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}}\, d\theta.$

Solution.
From Examples 6 and 7, we have

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\tan\theta}}\, d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) \times \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{1}{4}\left[\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)\right]^2$$

Show that $\int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \pi.$

Solution.
We know that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \frac{1}{2}\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$\therefore \quad \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta = \frac{1}{2}\beta\left(\frac{\frac{1}{2}+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sqrt{\sin\theta}\, d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\left(\frac{5}{4}-1\right)\Gamma\left(\frac{5}{4}-1\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)} = 2\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

and $$\int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta = \frac{1}{2}\beta\left(\frac{-\frac{1}{2}+1}{2}, \frac{1}{2}\right)$$

$$= \frac{1}{2}\beta\left(\frac{1}{4}, \frac{1}{2}\right) = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\therefore \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}}\, d\theta = 2\frac{\Gamma\left(\frac{3}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)}\frac{1}{2}\frac{\Gamma\left(\frac{1}{4}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \Gamma\left(\frac{1}{2}\right)\times\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\times\sqrt{\pi} = \pi$$

Prove that $\int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$, where a and n are positive.

Solution.

We know $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$. Let $I = \int_0^\infty e^{-ax} x^{n-1} dx$

Put $t = ax$ $\therefore \quad dt = a dx \Rightarrow dx = \frac{dt}{a}$

When $x = 0$, $t = 0$ and when $x = \infty$, $t = \infty$

$$\therefore \quad I = \int_0^\infty e^{-t} \left[\frac{t}{a}\right]^{n-1} \frac{dt}{a} = \frac{1}{a^n} \int_0^\infty e^{-t} t^{n-1} dt = \frac{\Gamma n}{a^n}$$

$$\Rightarrow \quad \int_0^\infty e^{-ax} x^{n-1} dx = \frac{\Gamma n}{a^n}$$

Show that $\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy,\ n > 0.$

Solution.

We have $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx,\ n > 0$

Put $y = e^{-x} \Rightarrow e^x = \frac{1}{y} \Rightarrow x = \log \frac{1}{y} \quad \therefore \quad dx = \frac{1}{\frac{1}{y}}\left(-\frac{1}{y^2}\right)dy = -\frac{1}{y}dy$

When $x = 0, y = 1$ and when $x = \infty, y = 0$

$$\therefore \quad \Gamma n = \int_1^0 y \left(\log \frac{1}{y}\right)^{n-1} \left(-\frac{1}{y}dy\right) = -\int_1^0 \left(\log \frac{1}{y}\right)^{n-1} dy = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

Prove that $\int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$.

Solution.

Let $$I = \int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

Let $$I_1 = \int_0^1 \frac{x^2dx}{\sqrt{1-x^4}} = \int_0^1 x^2(1-x^4)^{-\frac{1}{2}}\,dx.$$

Property (4) is $$\int_0^1 x^m(1-x^n)^p\,dx = \frac{1}{n}\frac{\Gamma(p+1)\,\Gamma\left(\frac{m+1}{n}\right)}{\Gamma\left(p+1+\frac{m+1}{n}\right)}$$

$\therefore$ $$I_1 = \int_0^1 x^2(1-x^4)^{-\frac{1}{2}}dx = \frac{1}{4}\frac{\Gamma\left(-\frac{1}{2}+1\right)\Gamma\left(\frac{2+1}{4}\right)}{\Gamma\left(-\frac{1}{2}+1+\frac{2+1}{4}\right)}$$ [Here $m=2,\ n=4,\ p=-\frac{1}{2}$]

$$= \frac{1}{4}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{1}{4}\sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{5}{4}\right)} = \frac{1}{4}\sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}$$

Let $I_2 = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

Put $x^2 = \tan\theta \quad \therefore \; 2x\,dx = \sec^2\theta\, d\theta \quad \Rightarrow \quad dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}}$

When $x = 0,\ \theta = 0$ and when $x = 1,\ \theta = \frac{\pi}{4}$

$$\therefore \qquad I_2 = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}\left(2\sqrt{\tan\theta}\right)} = \frac{1}{2}\int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec\theta\sqrt{\frac{\sin\theta}{\cos\theta}}}$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\sin\theta\cos\theta}}\, d\theta = \frac{\sqrt{2}}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin 2\theta}}$$

Let $2\theta = \phi, \qquad \therefore \; 2\,d\theta = d\phi \quad \Rightarrow \quad d\theta = \frac{d\phi}{2}$

Let $2\theta = \phi, \quad \therefore\ 2d\theta = d\phi \quad \Rightarrow \quad d\theta = \dfrac{d\phi}{2}$

When $\theta = 0,\ \phi = 0$ and when $\theta = \dfrac{\pi}{4}, \phi = \dfrac{\pi}{2}$

$$\therefore \quad I_2 = \frac{\sqrt{2}}{2}\cdot\int_0^{\pi/2} \sin^{-1/2}\phi\,\frac{d\phi}{2} = \frac{\sqrt{2}}{4}\int_0^{\pi/2}\sin^{-1/2}\phi\,d\phi$$

$$= \frac{\sqrt{2}}{4}\frac{1}{2}\beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right) \qquad \left[\because\ p = -\frac{1}{2},\ q = 0\right]$$

$$= \frac{\sqrt{2}}{8}\beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{\sqrt{2}}{8}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{\sqrt{2}}{8}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\sqrt{\pi}}{4\sqrt{2}}\cdot\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

$$\therefore \quad I = I_1 \times I_2 = \int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \sqrt{\pi}\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}\right)}\cdot\frac{\sqrt{\pi}}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)} = \frac{\pi}{4\sqrt{2}}$$

1) Evaluate $\int_0^\infty x^6 e^{-2x} dx$

Sol: Given that $\int_0^\infty x^6 e^{-2x} dx$

Put $2x = y$

$dx = \frac{dy}{2}$

Limits:

When x=0 then y=0

x= ∞ then y= ∞

$$\int_0^\infty x^6 e^{-2x} dx = \int_0^\infty e^{-y} \left(\frac{y}{2}\right)^6 \frac{dy}{2}$$

$$= \frac{1}{128}\int_0^\infty e^{-y} y^6 \, dy$$

Here n-1=6

n=6+1=7

$$= \frac{1}{128}\int_0^\infty e^{-y} y^{7-1} \, dy$$

$$= \frac{\Gamma(7)}{128}$$

$$= \frac{6!}{128} = \frac{45}{8}$$

2) Evaluate $\int_0^\infty x^4 e^{-x^2} dx$

Sol: Given that $\int_0^\infty x^4 e^{-x^2} dx$

Put $x^2 = y$

$2xdx = dy$

$dx = \frac{dy}{2x} = \frac{dy}{2\sqrt{y}}$

Limits:

When x=0 then y=0

x= ∞ then y= ∞

$$\int_0^\infty x^4 e^{-x^2} dx = \int_0^\infty e^{-y} (y^{1/2})^4 \frac{dy}{2\sqrt{y}}$$

$$= \frac{1}{2}\int_0^\infty e^{-y} y^{3/2} dy$$ Here n-1=3/2

n=3/2+1=5/2

$$= \frac{1}{2}\int_0^\infty e^{-y} y^{\frac{5}{2}-1} dy$$

$$= \frac{\Gamma(5/2)}{2}$$

$$= \frac{1}{2}.\frac{3}{2}.\frac{1}{2}.\ \Gamma\left(\frac{1}{2}\right) = \frac{3}{8}\sqrt{\pi}$$

3) Evaluate $\int_0^{\infty} x^3 3^{-x} dx$

Sol: Given that $\int_0^{\infty} x^3\, 3^{-x} dx$

Put $3^{-x} = e^{-y}$

$$-x\, log3 = -y$$

$$dx = \frac{dy}{\log 3}$$

Limits:

When x=0 then y=0

x= ∞ then y= ∞

$$\int_0^{\infty} x^3\, 3^{-x} dx = \frac{1}{\log 3}\int_0^{\infty} \frac{y^3}{(log3)^3} e^{-y} dy$$

$$= \frac{1}{(log3)^4} \int_0^\infty e^{-y} y^{4-1} dy$$

$$= \frac{\Gamma(4)}{(log3)^4}$$

$$= \frac{3!}{(log3)^4}$$

$$= \frac{6}{(log3)^4}$$

4) Evaluate $\int_0^1 x^3(log\frac{1}{x})^2dx$

Sol: Given that $\int_0^1 x^3(log\frac{1}{x})^2dx$

Put log$\frac{1}{x} = y$

$$x = e^{-y}$$

$$\text{d}x = -e^{-y}\,dy$$

Limits:

When x=0 then y=∞

x= 1 then y= 0

$$\int_0^1 x^3(log\frac{1}{x})^2dx = -\int_{\infty}^{0} e^{-3y}\,y^2e^{-y}dy$$

$$= \int_{0}^{\infty} e^{-4y} y^2 dy$$

Put $4y = t$

$dY = \frac{dt}{4}$

Limits:

When Y=0 then t=0

y= ∞ then t= ∞

$$\int_{0}^{\infty} y^2 e^{-4y} dy = \int_{0}^{\infty} e^{-t} (\frac{t}{4})^2 \frac{dt}{4}$$

$$=\frac{1}{64}\int_0^{\infty} e^{-t}t^2dy$$

Here n-1=2

n=2+1=3

$$=\frac{1}{64}\int_0^{\infty} e^{-t}t^{3-1}dy$$

$$=\frac{\Gamma(3)}{64}$$

$$=\frac{2!}{64}=\frac{1}{32}$$

5) Evaluate $\int_0^\infty a^{-bx^2} dx$

Sol: Given that $\int_0^\infty a^{-bx^2} dx$

$$\text{Put } a^{-bx^2} = e^{-y}$$

$$-bx^2 \log a = -y$$

$$x^2 = \frac{1}{b\, loga}\, \text{y}$$

$$\text{d}x = \frac{1}{\sqrt{b \log a}} \frac{1}{2\sqrt{y}}\, dy$$

Limits:

When x=0 then y=0 and x= ∞ then y=∞

$$\int_0^\infty a^{-bx^2} dx = \int_0^\infty e^{-y} \frac{1}{\sqrt{b \log a}} \frac{1}{2\sqrt{y}}\, dy$$

$$= \frac{1}{2\sqrt{b\log a}}\int_0^{\infty} e^{-y} y^{-1/2} dy$$ Here n-1=-1/2

n=-1/2+1=1/2

$$= \frac{1}{2\sqrt{b\log a}}\int_0^{\infty} e^{-y} y^{\frac{1}{2}-1} dy$$

$$= \frac{1}{2\sqrt{b\log a}}\Gamma(1/2)$$

$$= \frac{\sqrt{\pi}}{2\sqrt{b\log a}}$$

4. $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

solution:

Let $-\log x = y$

$\log x = -y$

$x = e^{-y}$

$dx = -e^{-y} dy$

$x = 0, \; y = -\log x \to \infty$

$x = 1, \; y = -\log 1 = 0.$

$\int_0^1 \frac{dx}{\sqrt{-\log x}} = \int_\infty^0 \frac{1}{\sqrt{y}} \cdot -e^{-y} dy$

$$= \int_0^\infty e^{-y} \cdot y^{-\frac{1}{2}} dy$$

$$= \int_0^\infty e^{-y} \cdot y^{\frac{1}{2}-1} dy$$

$$= \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

5. $$\int_0^1 x^m \left(\log \frac{1}{x}\right)^n dx = \frac{n!}{(m+1)^{n+1}}.$$

Evaluate the following in terms of Beta and Gamma functions

1. Evaluate $\int_0^\infty x^2 e^{-6x} dx$

2. Evaluate $\int_0^1 x^3 (log x)^2 dx$